

Hierarchy of Critical Exponents on Sierpinski fractal resistor networks

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Abstract

Using the S_3 -symmetry of Sierpinski fractal resistor networks we determine the current distribution as well as the multifractals spectrum of moments of current distribution by using the real space renormalization group technique based on ($[q/4] + 1$) independent Schure's invariant polynomials of inwards flowing currents.

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1 INTRODUCTION

The study of infinite sets of exponents which originated in the field of turbulence [1], has recently become the focus of attention in a number of fields involving fractals or scaling objects [2, 3], ranging from random resistor networks [4, 5, 6], dynamical systems, diffusion limited aggregates (DLA) [7], to localization. What is common to all of these different fields is that, one wants to characterize the properties of a "weight" or "measure" associated to different parts of a fractal object. Modelization of electrical transport properties for inhomogeneous and composite materials by random resistor networks have been the subject of many recent works, Also other physical phenomena such as diffusion problems can be formulated in terms of electrical problem. Distribution of currents (or voltage drops) on a percolating structure in the scaling region are multifractals, in the sense that different moments scale with different exponents, that is, if we consider a system of length L , then the q -moment of the current distribution :

$$M_q = \sum_r I_r^q \quad (1-1)$$

scales as L^{-D_q} , where D_q is by no means a simple function of q . Thus each moment scales with its own anomalous dimension. This phenomena is characteristic of multifractals distributions. Actually this set of exponents first appeared in the field of turbulence and has recently become focus of attention in a number different fields such as diffusion limited aggregation, dynamical system and random resistor networks as mentioned above. Here in this paper we study the multifractals structure of current distribution on Sierpinsky fractal, since as Kirkpatrick had suggested, the so called back-bone of the percolating random resistor networks could be modeled by a fractal structure and among the fractal objects, the n -simplex one is simplest to study the various physical problems from random walk [8, 9, 10, 11] to electrical one on it [5, 6, 12, 13].

Here by using the S_3 -symmetry of Sierpinski fractal resistor networks (see Fig. 1) together

with the minimization of the electrical power, we have been able to determine the current distribution in Sierpinsky fractals with decimation numbers $b = 2, 3, 4$, and 5 . Then, using the independent Shure's S_3 invariant polynomials, which is proved that the required number of independent Shure's S_3 -invariant polynomials of degree q is $[q/4] + 1$, with $[]$ indicating the greatest integer parts, we have derived the results of reference[5] for $b = 2$ up to $q = 12$ and we have calculated D_q up to $q = 22$ for $b = 2, 3, 4$ and 5 . The organization of the article is as follows:

In section 2, we give a brief description of Sierpinski fractals, then in section 3, using the S_3 -symmetry of Sierpinsky fractal resistor networks and minimization of electrical power we have determined the inward flowing current of subfractals. In section 4 we talk about the independent Shure's S_3 -symmetry invariant polynomials of input currents. Section 5 is about the moments current distributions and their multifractals spectrum where it contains the main results of this paper. The paper ends with a brief conclusion and 5 appendices.

2 Sierpinski Fractal

To obtain Sierpinski fractal with decimation number b , we choose a triangle and divide its sides into b parts and then draw all possible lines through the dividend points parallel to the side of the triangle. Next, having omitted every other inner triangle, we repeat this for the remaining triangles or for the subfractals of the next higher order. This way Sierpinski fractals are constructed. To calculate the fractal dimension, we label subfractals of order $(l + 1)$ in terms of partition of $(b - 1)$ into 3 positive integers λ_1 , λ_2 and λ_3 . Each partition represents a subfractal of order l and λ shows the distance of the corresponding subfractal

from the sides of triangle. As an illustrating example, we show in Figure 2 the method of labeling a Sierpinski fractal with decimation number $b = 3$. Obviously, the number of all possible partitions is equal to the distribution of $(b - 1)$ objects among three boxes, which is the same as the Bose-Einstein distribution of $(b - 1)$ identical bosons in 3 quantum states. This is equal to

$$c = \frac{(b + 1)!}{(b - 1)!2!}.$$

According to the following definition, the fractal dimension d_F of a self-similar object is

$$(N^r)^{d_F} = 1,$$

with N as the number of similar objects, up to translation and rotation. For self-similar fractals, N is equal to the number of subfractals. Therefore, we have $N = C^l$ and $r = b^{-l}$.

Hence $d_F = \frac{\ln C}{\ln b}$, or

$$d_F = \frac{\ln(b + 1)!/(b - 1)!2!}{\ln b}.$$

3 Determination of inward flowing current of subfractals

We denote the j -th inward flowing current of subfractal which corresponds to the partition λ_1, λ_2 and λ_3 by $I_{\lambda_1, \lambda_2, \lambda_3}(\lambda_1 + \delta_{1,j}, \lambda_2 + \delta_{2,j}, \lambda_3 + \delta_{3,j})$. Therefore, I_1, I_2 and I_3 can be denoted by

$$I_{b-1,0,0}(b, 0, 0), \quad I_{0,b-1,0}(0, b, 0) \quad \text{and} \quad I_{0,0,b-1}(0, 0, b).$$

In order to determine the inward flowing currents in terms of $I_j, j = 1, 2$ and 3 , besides using Kirchhoff's law at each node, we have to minimize the power of Sierpinski fractal, that is we minimize the following expression:

$$\begin{aligned} & \sum_{\text{over possible partitions}} \sum_{j=1}^3 I_{\lambda_1, \lambda_2, \lambda_3}^2(\lambda_1 + \delta_{1,j}, \lambda_2 + \delta_{2,j}, \lambda_3 + \delta_{3,j}) \\ & + \mu_{\lambda_1, \lambda_2, \lambda_3} \left(\sum_{j=1}^3 I_{\lambda_1, \lambda_2, \lambda_3}(\lambda_1 + \delta_{1,j}, \lambda_2 + \delta_{2,j}, \lambda_3 + \delta_{3,j}) \right) \\ & + 2 \sum_{\text{over all nodes}} \nu_{\eta_1, \eta_2, \eta_3} \left(\sum_{j=1}^3 I_{\eta_1, \eta_2, \eta_3}(\eta_1 - \delta_{1,j}, \eta_2 - \delta_{2,j}, \eta_3 - \delta_{3,j}) \right). \end{aligned} \quad (3-1)$$

where $\mu_{\lambda_1, \lambda_2, \lambda_3}$ and $\nu_{\eta_1, \eta_2, \eta_3}$ are Lagrange multipliers which are considered because of Kirchhoff's law on each subfractal, and also each node, respectively. By minimizing the energy given by expression (1-2), we get the following equations for the inner flowing currents:

$$I_{\lambda_1, \lambda_2, \lambda_3}(\lambda_1 + \delta_{1,j}, \lambda_2 + \delta_{2,j}, \lambda_3 + \delta_{3,j}) + \mu_{\lambda_1, \lambda_2, \lambda_3} + n u_{\lambda_1 + \delta_{1,j}, \lambda_2 + \delta_{2,j}, \lambda_3 + \delta_{3,j}} = 0, \quad (3-2)$$

together with the Kirchhoff's law for each subfractal and each vertex, respectively:

$$\sum_{j=1}^3 I_{\lambda_1, \lambda_2, \lambda_3}(\lambda_1 + \delta_{1,j}, \lambda_2 + \delta_{2,j}, \lambda_3 + \delta_{3,j}) = 0, \quad (3-3)$$

$$\sum_{j=1}^3 I_{\eta_1, \eta_2, \eta_3}(\eta_1 - \delta_{1,j}, \eta_2 - \delta_{2,j}, \eta_3 - \delta_{3,j}) = 0. \quad (3-4)$$

Now, using the S_3 -permutation symmetry of Sierpinski fractal we propose the following ansatz for the Lagrange multipliers:

$$\mu_{\lambda_1, \lambda_2, \lambda_3} = - \sum_{k=1}^3 a_{\lambda_k} I_k, \quad (3-5)$$

$$\nu_{\lambda_1, \lambda_2, \lambda_3} = - \sum_{k=1}^3 b_{\lambda_k} I_k, \quad (3-6)$$

where a_0 is assumed to be zero. Substituting the ansatz (3-5) and (3-6) in equation (3-2), then the inward flowing currents can be given in terms of a's and b,s , respectively.

$$I_{\lambda_1, \lambda_2, \lambda_3}(\eta_1, \eta_2, \eta_3) = \sum_{k=1}^3 (a_{\lambda_k} + b_{\eta_k}) I_k. \quad (3-7)$$

Actually one could write the currents in terms of input ones as in (3-7) simply by using the symmetry of simplex fractal where the minimization of power is not required. Finally the a's and b's themselves can be determined through the equations (3-3) and (3-4). Here we determine the currents for b=2,3,4 and 5, respectively. First for b=2 we have

$$I_{1,0,0}(2, 0, 0) = I_1, \quad I_{0,1,0}(0, 2, 0) = I_2, \quad I_{0,0,1}(0, 0, 2) = I_3,$$

$$I_{\delta_{1,j}, \delta_{2,j}, \delta_{3,j}}(\delta_{1,j} + \delta_{1,k}, \delta_{2,j} + \delta_{2,k}, \delta_{3,j} + \delta_{3,k}) = a_1 I_J + b_1 I_j + b_1 I_k$$

Using equation (3-4) we obtain:

$$a_1 + 2b_1 = 0$$

and using equation (3-4) we get:

$$1 + 2a_1 + b_1 = 0.$$

Solving the above equations we get the following result:

$$I_{\delta_{1,j}, \delta_{2,j}, \delta_{3,j}}(\delta_{1j} + \delta_{1k}, \delta_{2j} + \delta_{2k}, \delta_{3j} + \delta_{3k}) = \frac{(I_k - I_j)}{3}.$$

Similarly for b=3 we have

$$I_{2,0,0}(3, 0, 0) = I_1, \quad I_{0,2,0}(0, 3, 0) = I_2, \quad I_{0,0,2}(0, 0, 3) = I_3,$$

$$I_{2\delta_{1,j}, 2\delta_{2,j}, 2\delta_{3,j}}(2\delta_{1,j} + \delta_{1,k}, 2\delta_{2,j} + \delta_{2,k}, 2\delta_{3,j} + \delta_{3,k}) = a_2 I_j + b_2 I_j + b_1 I_k,$$

$$I_{\delta_{1,j} + \delta_{1,k}, \delta_{2,j} + \delta_{2,k}, \delta_{3,j} + \delta_{3,k}}(2\delta_{1,j} + \delta_{1,k}, 2\delta_{2,j} + \delta_{2,k}, 2\delta_{3,j} + \delta_{3,k}) = a_1 I_j + a_1 I_k + b_2 I_j + b_1 I_k,$$

$$\begin{aligned} & I_{\delta_{1,j} + \delta_{1,k}, \delta_{2,j} + \delta_{2,k}, \delta_{3,j} + \delta_{3,k}}(\delta_{1,j} + \delta_{1,k} + \delta_{1,l}, \delta_{2,j} + \delta_{2,k} + \delta_{2,l}, \delta_{3,j} + \delta_{3,k} + \delta_{3,l}) \\ &= a_1(I_j + I_k) + b_1(I_j + I_k + I_l). \end{aligned}$$

Using equation (3-3) in subfractal $(2\delta_{1,j}, 2\delta_{2,j}, 2\delta_{3,j})$, we get

$$1 + 2(a_2 + b_2) - b_1 = 0.$$

Also using equation (3-3) in subfractal $(\delta_{1,j} + \delta_{1,K}, \delta_{2,j} + \delta_{2,K}, \delta_{3,j} + \delta_{3,K})$ we get

$$3a_1 + 2b_1 + b_2 = 0.$$

Also using equation (3-4) in the vertices we have

$$a_1 + a_2 + 2b_1 = 0,$$

$$a_1 + 2b_1 = 0,$$

$$2a_1 + 3b_1 = 0.$$

After solving the above equations we get the following result for the currents for $b=3$

$$I_{2\delta_{1,j}, 2\delta_{2,j}, 2\delta_{3,j}}(2\delta_{1,j} + \delta_{1,k}, 2\delta_{2,j} + \delta_{2,k}, 2\delta_{3,j} + \delta_{3,k}) = -\frac{9}{21} I_j + \frac{3}{21} I_k,$$

$$\begin{aligned} I_{\delta_{1,j}+\delta_{1,k}, \delta_{2,j}+\delta_{2,k}, \delta_{3,j}+\delta_{3,k}}(2\delta_{1,j} + \delta_{1,k}, 2\delta_{2,j} + \delta_{2,k}, 2\delta_{3,j} + \delta_{3,k}) \\ = \frac{9}{21}I_j - \frac{3}{21}I_k, \end{aligned}$$

$$\begin{aligned} I_{\delta_{1,j}+\delta_{1,k}, \delta_{2,j}+\delta_{2,k}, \delta_{3,j}+\delta_{3,k}}(\delta_{1,j} + \delta_{1,k} + \delta_{1,l}, \delta_{2,j} + \delta_{2,k} + \delta_{2,l}, \delta_{3,j} + \delta_{3,k} + \delta_{3,l}) \\ = -\frac{6}{21}(I_j + I_k) + \frac{4}{21}(I_j + I_k + I_l). \end{aligned}$$

By the same procedure explained above, we can calculate the inner inward flowing currents for decimation number $b = 4$ and $b = 5$, where we quote only the results below and give the details of calculation in Appendix I and II.

3.1 Inner inward flowing currents corresponding to $b = 4$:

$$I_{3\delta_{1,j}, 3\delta_{2,j}, 3\delta_{3,j}}(3\delta_{1,j} + \delta_{1,k}, 3\delta_{2,j} + \delta_{2,k}, 3\delta_{3,j} + \delta_{3,k}) = -\frac{19}{41}I_j + \frac{3}{41}I_k,$$

$$I_{2\delta_{1,j}+\delta_{1,k}, 2\delta_{2,j}+\delta_{2,k}, 2\delta_{3,j}+\delta_{3,k}}(3\delta_{1,j} + \delta_{1,k}, \delta_{2,j} + \delta_{2,k}, 3\delta_{3,j} + \delta_{3,k}) = \frac{19}{41}I_j - \frac{3}{41}I_k,$$

$$\begin{aligned} I_{2\delta_{1,j}+\delta_{1,k}, 2\delta_{2,j}+\delta_{2,k}, 2\delta_{3,j}+\delta_{3,k}}(2\delta_{1,j} + 2\delta_{1,k}, 2\delta_{2,j} + 2\delta_{2,k}, 2\delta_{3,j} + 2\delta_{3,k}) \\ = -\frac{9}{41}(I_j - I_k), \end{aligned}$$

$$\begin{aligned} I_{2\delta_{1,j}+\delta_{1,k}, 2\delta_{2,j}+\delta_{2,k}, 2\delta_{3,j}+\delta_{3,k}}(2\delta_{1,j} + \delta_{1,k} + \delta_{1,l}, 2\delta_{2,j} + \delta_{2,k} + \delta_{2,l}, 2\delta_{3,j} + \delta_{3,k} + \delta_{3,l}) \\ = -\frac{184}{1353}I_j - \frac{52}{1353}I_k + \frac{146}{1353}I_l, \end{aligned}$$

$$I_{\delta_{1,j}+\delta_{1,k}+\delta_{1,l}, \delta_{2,j}+\delta_{2,k}+\delta_{2,l}, \delta_{3,j}+\delta_{3,k}+\delta_{3,l}}(2\delta_{1,j} + \delta_{1,k} + \delta_{1,l}, 2\delta_{2,j} + \delta_{2,k} + \delta_{2,l}, 2\delta_{3,j} + \delta_{3,k} + \delta_{3,l})$$

$$= \frac{368}{1353}I_j - \frac{94}{1353}(I_k + I_l).$$

3.2 Inner inward flowing currents corresponding to $b = 5$:

$$I_{4\delta_{1,j}, 4\delta_{2,j}, 4\delta_{3,j}}(4\delta_{1,j} + \delta_{1,k}, 4\delta_{2,j} + \delta_{2,k}, 4\delta_{3,j} + \delta_{3,k}) = \frac{283}{591}I_j - \frac{41375}{1015929}I_k,$$

$$\begin{aligned} I_{3\delta_{1,j} + \delta_{1,k}, 3\delta_{2,j} + \delta_{2,k}, 3\delta_{3,j} + \delta_{3,k}}(4\delta_{1,j} + \delta_{1,k}, 4\delta_{2,j} + \delta_{2,k}, 4\delta_{3,j} + \delta_{3,k}) \\ = -\frac{283}{591}I_j + \frac{41375}{1015929}I_k, \end{aligned}$$

$$\begin{aligned} I_{3\delta_{1,j} + \delta_{1,k}, 3\delta_{2,j} + \delta_{2,k}, 3\delta_{3,j} + \delta_{3,k}}(3\delta_{1,j} + 2\delta_{1,k}, 3\delta_{2,j} + 2\delta_{2,k}, 3\delta_{3,j} + 2\delta_{3,k}) \\ = \frac{51}{197}I_j - \frac{25}{197}I_k, \end{aligned}$$

$$\begin{aligned} I_{3\delta_{1,j} + \delta_{1,k}, 3\delta_{2,j} + \delta_{2,k}, 3\delta_{3,j} + \delta_{3,k}}(3\delta_{1,j} + \delta_{1,k} + \delta_{1,l}, 3\delta_{2,j} + \delta_{2,k} + \delta_{2,l}, 3\delta_{3,j} + \delta_{3,k} + \delta_{3,l}) \\ = 2\frac{17486}{112881}I_j + 2\frac{2206}{112881}I_k - 2\frac{2448}{37627}I_l, \end{aligned}$$

$$\begin{aligned} I_{2\delta_{1,j} + 2\delta_{1,k}, 2\delta_{2,j} + 2\delta_{2,k}, 2\delta_{3,j} + 2\delta_{3,k}}(3\delta_{1,j} + 2\delta_{1,k}, 3\delta_{2,j} + 2\delta_{2,k}, 3\delta_{3,j} + 2\delta_{3,k}) \\ = -\frac{51}{197}I_j - \frac{25}{197}I_k, \end{aligned}$$

$$\begin{aligned} I_{2\delta_{1,j} + 2\delta_{1,k}, 2\delta_{2,j} + 2\delta_{2,k}, 2\delta_{3,j} + 2\delta_{3,k}}(2\delta_{1,j} + 2\delta_{1,k} + \delta_{1,l}, 2\delta_{2,j} + 2\delta_{2,k} + \delta_{2,l}, 2\delta_{3,j} + 2\delta_{3,k} + \delta_{3,l}) \\ = \frac{9865}{338643}(2I_j + I_k) - \frac{12482}{338643}I_l \end{aligned}$$

$$\begin{aligned} I_{2\delta_{1,j} + \delta_{1,k} + \delta_{1,l}, 2\delta_{2,j} + \delta_{2,k} + \delta_{2,l}, 2\delta_{3,j} + \delta_{3,k} + \delta_{3,l}}(3\delta_{1,j} + \delta_{1,k} + \delta_{1,l}, 3\delta_{2,j} + \delta_{2,k} + \delta_{2,l}, 3\delta_{3,j} + \delta_{3,k} + \delta_{3,l}) \\ = 2\frac{5138}{112881}(I_k + I_l) - 4\frac{34972}{112881}I_j, \end{aligned}$$

$$I_{2\delta_{1,j} + \delta_{1,k} + \delta_{1,l}, 2\delta_{2,j} + \delta_{2,k} + \delta_{2,l}, 2\delta_{3,j} + \delta_{3,k} + \delta_{3,l}}(2\delta_{1,j} + 2\delta_{1,k} + \delta_{1,l}, 2\delta_{2,j} + 2\delta_{2,k} + \delta_{2,l}, 2\delta_{3,j} + 2\delta_{3,k} + \delta_{3,l})$$

$$= 2 \frac{18847}{3047787} I_j - 2 \frac{14356}{3047787} I_k + \frac{624}{3047787} I_l.$$

4 Shure's Polynomials of Inward Flowing Currents

Shure's S_3 -invariant polynomials are homogeneous polynomials of degree 3 of variables I_1, I_2 and I_3 :

$$S_{\lambda_1, \lambda_2, \lambda_3} = \sum_{\text{permutation of } (1,2,3)} I_1^{\lambda_1} I_2^{\lambda_2} I_3^{\lambda_3}$$

where $\lambda_1, \lambda_2, \lambda_3$ are partitions of m into 3 non-negative integers, that is:

$$\lambda_1 + \lambda_2 + \lambda_3 = m.$$

Because of the following equation due to Kirchhoff's law:

$$S_1 = \sum_{k=1}^3 I_k = 0, \quad (4-1)$$

all Schure's polynomials of degree m , corresponding to all possible partitions of m , are not independent. In calculation of the multifractals critical exponents D_q , we must use the independent ones. By multiplying both sides of (4-2) by $S_{\lambda_1, \lambda_2, \lambda_3}$, we get

$$0 = S_1 S_{\lambda_1, \lambda_2, \lambda_3} = \sum a_{\mu_1, \mu_2, \mu_3} S_{\mu_1, \mu_2, \mu_3}, \quad (4-2)$$

where (μ_1, μ_2, μ_3) and $(\lambda_1, \lambda_2, \lambda_3)$ correspond to partition of $m-1$ and m respectively. From the formula (2-10) it follows that there are $P_3(m+1)$ constraint over $P_3(m)$ shure polynomials of degree m , where $p_3(m)$ takes all possible partitions of m into 3 non-negative integers. Therefore, the number of invariant polynomials of degree m is:

$$P_3(m) - P_3(m-1). \quad (4-3)$$

For example for $m = 2$ we have

$$0 = S_1 S_1 = S_2 + 2S_{1,1}$$

therefore, using the above equation we can write $S_{1,1}$ in terms of S_2 as:

$$S_{1,1} = -\frac{S_2}{2}. \quad (4-4)$$

Thus we have only one invariant polynomial for $q = 2$. Also in the case of $q = 4$ we have

$$S_1 S_3 = S_4 + S_{3,1} = 0,$$

$$S_1 S_{2,1} = S_{3,1} + 2S_{2,2} + S_{2,1,1} = 0,$$

$$S_1 S_{1,1,1} = S_{2,1,1} = 0,$$

hence there is only one independent polynomial such as S_4 and the others can be written in terms of S_4 as follows:

$$\begin{aligned} S_{3,1} &= -S_4, & S_{2,2} &= \frac{S_4}{2}, \\ S_{2,1,1} &= 0. \end{aligned} \quad (4-5)$$

For $q = 6$ we have:

$$S_1 S_5 = S_6 + S_{5,1} = 0,$$

$$S_1 S_{4,1} = S_{5,1} + S_{4,2} + S_{4,1,1} = 0,$$

$$S_1 S_{3,2} = S_{4,2} + 2S_{3,3} + S_{3,2,1} = 0,$$

$$S_1 S_{3,1,1} = S_{4,1,1} + S_{3,2,1} = 0.$$

Therefore, there are only two independent invariant polynomials such as S_6 , $S_{3,3}$ and the other dependent one can be written in terms of them as follows:

$$S_{5,1} = -S_6, \quad S_{4,2} = \frac{S_6}{2} - S_{3,3}, \quad S_{321} = -\frac{S_6}{2} - S_{3,3}, \quad S_{4,1,1} = \frac{S_6}{2} + S_{3,3}$$

In Appendix III, we have proved that the number of independent Schure's invariant polynomials of degree q is equal to:

$$[q/4] + 1 \quad (4-6)$$

where $[]$ means the greatest integer part.

Below we give some of the constraints over Schure's invariant polynomials of degrees $q = 8$ and 10 which are occurring through imposing the Kirschhoff's law over Schure's polynomials of order eight:

$$S_8 + S_{7,1} = 0, \quad S_{7,1} + S_{6,2} + 2S_{6,1,1} = 0,$$

$$S_{6,2} + S_{5,3} + S_{5,2,1} = 0, \quad S_{6,1,1} + S_{5,2,1} = 0,$$

$$S_{5,3} + 2S_{4,4} + S_{4,3,1} = 0, \quad S_{5,2,1} + S_{4,3,1} + 2S_{4,2,2} = 0,$$

where, S_8 and $S_{4,2,2}$ are considered as the invariant polynomials and other dependent invariant can be expressed in terms of them as follows:

$$S_{7,1} = -S_8, \quad S_{3,3,2} = S_{4,2,2},$$

$$S_{4,3,1} = -S_{4,2,2}, \quad S_{5,2,1} = 5S_{4,2,2},$$

$$S_{6,1,1} = -5S_{4,2,2}, \quad S_{6,2} = S_8 + 10S_{4,2,2},$$

$$S_{5,3} = -15S_{4,2,2} - S_8, \quad S_{4,4} = \frac{S_8 + 16S_{4,2,2}}{2}.$$

Constraints over Schure's polynomials of order ten are:

$$S_{10} + S_{9,1} = 0, \quad S_{8,2} + S_{9,1} + 2S_{8,1,1} = 0,$$

$$S_{8,2} + S_{7,3} + S_{7,2,1} = 0, \quad S_{8,1,1} + S_{7,2,1} = 0,$$

$$S_{7,3} + S_{6,4} + S_{6,3,1} = 0, \quad S_{7,2,1} + S_{6,3,1} + 2S_{6,2,2},$$

$$S_{6,4} + 2S_{5,5} + S_{5,4,1} = 0 \quad S_{6,3,1} + S_{5,4,1} + S_{5,3,2} = 0,$$

$$S_{6,2,2} + S_{5,3,2} = 0$$

where S_{10} and $S_{4,4,2}$ are considered as the invariant polynomials and other dependent invariants can be expressed in terms of them as follows:

$$S_{4,3,3} = 0, \quad S_{5,3,2} = -2S_{4,4,2},$$

$$S_{6,2,2} = 2S_{4,4,2}, \quad S_{9,1} = -S_{10},$$

$$S_{5,4,1} = -S_{4,4,2}, \quad S_{6,3,1} = 3S_{4,4,2},$$

$$S_{7,2,1} = -7S_{4,4,2}, \quad S_{8,1,1} = 7S_{4,4,2},$$

$$S_{8,2} = S_{10} - 14S_{4,4,2}, \quad S_{7,3} = 21S_{4,4,2} - S_{10},$$

$$S_{6,4} = S_{10} - 24S_{4,4,2}, \quad S_{5,5} = \frac{25S_{4,4,2} - S_{10}}{2},$$

In Appendix IV we use the constraints concerned with the invariant polynomials of order up to 22 to express the dependent invariant polynomials in terms of the independent ones.

5 Moments of Current Distribution and Multifractal Spectrum

In order to study the multifractals behaviour of current distribution we consider their q -moments defined as:

$$M_q(n) = \sum_r I_r(n)^q$$

where I_r is the current in the r -th bond of subfractals of generation level n . From the S_3 symmetry of Sierpinsky fractal, it is clear that the q -moments depend only on the independent Schure's S_3 invariant polynomials of degree q of input currents defined in section IV, that is

$$M_q(n+1) = \sum_{\substack{\text{partitions} \\ \text{corresponding} \\ \text{to} \\ \text{independent} \\ \text{polynomials}}} A_{\lambda_1, \lambda_2, \lambda_3}(n+1) S_{\lambda_1, \lambda_2, \lambda_3}(n+1), \quad (5-1)$$

where $A_{\lambda_1, \lambda_2, \lambda_3}$ s are some constants.

On the other hand, $M_q(n+1)$ can be written in terms of the invariant polynomials of their level n subfractals, that is

$$M_q(n+1) = \sum_{\substack{\text{partitions} \\ \text{corresponding} \\ \text{to} \\ \text{invariant} \\ \text{polynomials}}} A_{\lambda_1, \lambda_2, \lambda_3}(n) S_{\lambda_1, \lambda_2, \lambda_3}(n). \quad (5-2)$$

By comparing the expressions (5-1) and (5-2) we obtain the recursion relations between $A_{\lambda_1, \lambda_2, \lambda_3}(n)$ and $A_{\lambda_1, \lambda_2, \lambda_3}(n+1)$. Then the scaling factor is defined as:

$$\lambda(q) = \lim_{n \rightarrow \infty} \frac{M_q(n+1)}{M_q(n)}. \quad (5-3)$$

Obviously $\lambda(q)$ is the maximum eigenvalue of the matrix connecting $A(n)$ and $A(n+1)$. Then $D(q)$, the multifractals scaling exponents, are defined as:

$$D(q) = \frac{\ln(\lambda(q))}{\ln(b)}, \quad (5-4)$$

since the $M_q(n)$ scale as:

$$\lim_{n \rightarrow \infty} M_q(n) = L_n^{D(q)},$$

where $L_n = b^n$.

Now, as an example, we obtain D_2 , the power scaling exponent of Sierpinsky fractal with decimation numbers $b = 2, 3, 4, 4$ and 5 .

According to formula (4-4) for $q = 2$ we have only one independent invariant polynomial, where we can consider S_2 as the independent invariant polynomial. Therefore the total power is proportional to S_2 , that is:

$$P(n+1) = A_2(n+1)S_2(n+1). \quad (5-5)$$

It is straightforward to show that:

$$\begin{aligned} \frac{A_2(n+1)}{A_2(n)} &= \frac{5}{3} \quad \text{for } b = 2 \\ \frac{A_2(n+1)}{A_2(n)} &= \frac{45}{21} \quad \text{for } b = 3 \\ \frac{A_2(n+1)}{A_2(n)} &= \frac{3399}{1353} \quad \text{for } b = 4 \\ \frac{A_2(n+1)}{A_2(n)} &= \frac{8576091}{3047787} \quad \text{for } b = 5 \end{aligned}$$

Therefore, we have:

$$D(2) = .7369655945 \quad \text{for } b = 2$$

$$D(2) = .6937297714 \quad \text{for } b = 3$$

$$D(2) = .6644742613 \quad \text{for } b = 4$$

$$D(2) = .6428097998 \quad \text{for } b = 5$$

In case of $D(4)$, we have to consider the $S_4, S_{3,1}, S_{2,2}, S_{2,1,1}$, and due to relations (4-5) only one of them, say S_4 , is independent and the others can be written in terms of it. Again by computing the fourth moments of currents of fractals which are proportional to S_4 :

$$M_4(n) = A_4(n)S_4(n) \quad M_4(n+1) = A_4(n+1)S_4(n+1) \quad (5-6)$$

one can easily show that:

$$\begin{aligned} \frac{A_{4(n+1)}}{A_{4(n)}} &= 1.222222222 \quad \text{for } b = 2 \\ \frac{A_{4(n+1)}}{A_{4(n)}} &= 1.288213244 \quad \text{for } b = 3 \\ \frac{A_{4(n+1)}}{A_{4(n)}} &= 1.323683604 \quad \text{for } b = 4 \\ \frac{A_{4(n+1)}}{A_{4(n)}} &= 1.231193828 \quad \text{for } b = 5. \end{aligned}$$

Hence using the formula (5-4) we get

$$D(4) = .2895066169 \quad \text{for } b = 2$$

$$D(4) = .2305237058 \quad \text{for } b = 3$$

$$D(4) = .2022791602 \quad \text{for } b = 4$$

$$D(4) = .1292279056 \quad \text{for } b = 5$$

With the above explained prescription, we can calculate the higher moments and consequently higher multifractals exponents, where we quote only the multifractals exponents below in the remaining part of this section and give the other information such as the recursion relations in appendix V:

Table 1: The multifractals scaling exponents for $b = 2, 3, 4, 5$ and $q = 6, 8, 10, 12, 14, 16, 18, 20, 22$

	b=2	b=3	b=4	b=5
$D(q = 6)$	0.08779681671	.06123675596	.04899319186	.02514677379
$D(q = 8)$.02283703573	.01502761610	.01395480248	.006252791011
$D(q = 10)$.005703291923	.003671203776	.002879616683	.001058598646
$D(q = 12)$.001418367685	.0009335494363	.0007112996346	.0002227018237
$D(q = 14)$.0003533794924	.0002242180076	.0001768548943	.00006087603594
$D(q = 16)$.00008819069499	.00005581142757	.00004410760965	.00001057440927
$D(q = 18)$.00002202978507	.00001392382295	.00001101561388	.000001496172722
$D(q = 20)$.000005503871083	.000003476196968	.000002732459233	.00000008760821964
$D(q = 22)$.000001376330413	.000002165456544	.0000006881652063	.00000005343480209

Using the above results, the best fit we can get for various multifractals exponents are:

$$D(q, b = 2) = 1 + 4 \times 2^{-q},$$

$$D(q, b = 3) = 1 + 51.47353178 \times 3^{-q},$$

$$D(q, b = 4) = 1 + 291.7913871 \times 4^{-q},$$

$$D(q, b = 5) = 1 + 650.6706017 \times 5^{-q},$$

where the first formula is the same as the formula of reference[5]. The above formulas show the scaling behaviour of the multifractals spectra.

Appendix I: Calculation of currents of $b = 4$.

Here in this Appendix we give the detail of the calculation of inner inward flowing currents corresponding to decimation number **b=4**. Following the procedure of section III, for **b=4**

we have:

$$I_{3\delta_{1,j}, 3\delta_{2,j}, 3\delta_{3,j}}(4\delta_{1,j}, 4\delta_{2,j}, 4\delta_{3,j}) = I_j,$$

$$I_{3\delta_{1,j}, 3\delta_{2,j}, 3\delta_{3,j}}(3\delta_{1,j} + \delta_{1,k}, 3\delta_{2,j} + \delta_{2,k}, 3\delta_{3,j} + \delta_{3,k}) = a_3(3)I_j + b_{31}(3)I_j + b_{31}(1)I_k,$$

$$I_{2\delta_{1,j} + \delta_{1,k}, 2\delta_{2,j} + \delta_{2,k}, 2\delta_{3,j} + \delta_{3,k}}(3\delta_{1,j} + \delta_{1,k}, 3\delta_{2,j} + \delta_{2,k}, 3\delta_{3,j} + \delta_{3,k}) = a_{21}(2)I_j + a_{21}(1)I_k + b_{31}(3)I_j + b_{31}(1)I_k,$$

$$I_{2\delta_{1,j} + \delta_{1,k}, 2\delta_{2,j} + \delta_{2,k}, 2\delta_{3,j} + \delta_{3,k}}(2\delta_{1,j} + 2\delta_{1,k}, 2\delta_{2,j} + 2\delta_{2,k}, 2\delta_{3,j} + 2\delta_{3,k}) = a_{21}(2)I_j + a_{21}(1)I_k + b_{22}(2)(I_j + I_k),$$

$$\begin{aligned} & I_{2\delta_{1,j} + \delta_{1,k}, 2\delta_{2,j} + \delta_{2,k}, 2\delta_{3,j} + \delta_{3,k}}(2\delta_{1,j} + \delta_{1,k} + \delta_{1,l}, 2\delta_{2,j} + \delta_{2,k} + \delta_{2,l}, 2\delta_{3,j} + \delta_{3,k} + \delta_{3,l}) \\ &= a_{21}(2)I_j + a_{21}(1) + b_{211}(2)I_j + b_{211}(1)(I_k + I_l), \end{aligned}$$

$$\begin{aligned} & I_{\delta_{1,j} + \delta_{1,k} + \delta_{1,l}, \delta_{2,j} + \delta_{2,k} + \delta_{2,l}, \delta_{3,j} + \delta_{3,k} + \delta_{3,l}}(2\delta_{1,j} + \delta_{1,k} + \delta_{1,l}, 2\delta_{2,j} + \delta_{2,k} + \delta_{2,l}, 2\delta_{3,j} + \delta_{3,k} + \delta_{3,l}) \\ &= -47a_{111}(1)(I_j + I_k + I_l) + b_{211}(2)I_j + b_{211}(1)(I_k + I_l). \end{aligned}$$

Now, imposing Kirchhoff's law on subfractals and on vertices, we get the following equations for **a** and **b**

$$1 + 2a_3(3) + b_{31}(3) - b_{31}(1) = 0,$$

$$3a_{21}(2) + b_{31}(3) + b_{22}(2) + 2b_{211}(2) - b_{211}(1) = 0,$$

$$3a_{21}(2) + b_{31}(1) + b_{22}(2) + b_{211}(1) = 0,$$

$$3a_{111}(2) + b_{211}(2) = 0$$

$$a_{21}(1) + 2b_{31}(1) = 0,$$

$$a_3(3) + b_{21}(2) + 2b_{31}(3) = 0,$$

$$a_{21}(2) + a_{21}(1) + 2b_{22}(2) = 0,$$

$$2a_{21}(2) + a_{111}(1) + 3b_{211}(2) = 0,$$

$$a_{21}(1) + a_{111}(1) + 3b_{211}(1) = 0,$$

$$3a_{111}(1) + 4b_{1111}(1) = 0.$$

By solving the above equations we can determine inner inward flowing currents corresponding to decimation number **b=4** which is given in subsection 3.1.

Appendix II: Calculation of currents of **b=5**.

Here in this Appendix we give the detail of the calculation of inner inward flowing currents corresponding to decimation number **b=5**. Following the procedure of section III, for **b=5** we have:

$$I_{4\delta_{1,j},4\delta_{2,j},4\delta_{3,j}}(5\delta_{1,j}, 5\delta_{2,j}, 5\delta_{3,j}) = I_j,$$

$$I_{4\delta_{1,j},4\delta_{2,j},4\delta_{3,j}}(4\delta_{1,j} + \delta_{1,k}, 4\delta_{2,j} + \delta_{2,k}, 4\delta_{3,j} + \delta_{3,k}) = a_4(4)I_j + b_{41}(4)I_j + b_{41}(1)I_k,$$

$$I_{3\delta_{1,j}+\delta_{1,k},3\delta_{2,j}+\delta_{2,k},3\delta_{3,j}+\delta_{3,k}}(4\delta_{1,j}+\delta_{1,k}, 4\delta_{2,j}+\delta_{2,k}, 4\delta_{3,j}+\delta_{3,k}) = a_{31}(3)I_j + a_{31}(1)I_k + b_{41}(4)I_j + b_{41}(1)I_k,$$

$$\begin{aligned}
& I_{3\delta_{1,j}+\delta_{1,k}, 3\delta_{2,j}+\delta_{2,k}, 3\delta_{3,j}+\delta_{3,k}}(3\delta_{1,j}+2\delta_{1,k}, 3\delta_{2,j}+2\delta_{2,k}, 3\delta_{3,j}+2\delta_{3,k}) \\
& = a_{31}(3)I_j + a_{31}(1)I_k + b_{32}(3)I_j + b_{32}(3)I_k, \\
& I_{3\delta_{1,j}+\delta_{1,k}, 3\delta_{2,j}+\delta_{2,k}, 3\delta_{3,j}+\delta_{3,k}}(3\delta_{1,j}+\delta_{1,k}+\delta_{1,l}, 3\delta_{2,j}+\delta_{2,k}+\delta_{2,l}, 3\delta_{3,j}+\delta_{3,k}+\delta_{3,l}) \\
& = a_{31}(3)I_j + a_{31}(1) + b_{311}(3)I_j + b_{311}(1)(I_k + I_l),
\end{aligned}$$

$$I_{2\delta_{1,j}+2\delta_{1,k}, 2\delta_{2,j}+2\delta_{2,k}, 2\delta_{3,j}+2\delta_{3,k}}(3\delta_{1,j}+2\delta_{1,k}, 3\delta_{2,j}+2\delta_{2,k}, 3\delta_{3,j}+2\delta_{3,k}) = a_{22}(2)(I_j + I_k) + b_{32}(3)I_j + b_{32}(2)I_k.$$

Again imposing Kirchhoff's law on subfractals and on vertices, we get the following equations for **a** and **b**:

$$\begin{aligned}
& 1 + 2a_4(4) + b_{41}(3) - b_{41}(1) = 0, \\
& 3a_{31}(3) + b_{41}(4) + b_{32}(3) + b_{311}(3) - b_{311}(1) = 0, \\
& 3a_{31}(1) + b_{41}(1) + b_{32}(2) = 0, \\
& 3a_{22}(1) + b_{32}(3) + b_{32}(2) + b_{221}(2) - b_{221}(1) = 0, \\
& 3a_{211}(2) + b_{311}(3) + 2b_{221}(2) - b_{2111}(1) = 0, \\
& 3a_{211}(2) + b_{311}(1) + b_{221}(2) + b_{221}(1) = 0, \\
& 3a_{1111}(2) + b_{2111}(2) + 3b_{2111}(1) = 0, \\
& a_4(4) + a_{31}(3) + 2b_{41}(4) = 0, \\
& a_{31}(1) + 2b_{41}(1) = 0, \\
& a_{31}(3) + a_{22}(2) + 2b_{32}(3) = 0, \\
& a_{31}(1) + a_{22}(2) + 2b_{32}(2) = 0, \\
& 2a_{31}(3) + a_{211}(2) + 3b_{311}(3) = 0,
\end{aligned}$$

$$\begin{aligned}
a_{31}(1) + a_{211}(1) + 3b_{311}(1) &= 0, \\
a_{22}(2) + a_{211}(2) + a_{211}(1) + 3b_{221}(2) &= 0, \\
2a_{211}(1) + 3b_{221}(1) &= 0, \\
3a_{211}(2) + a_{1111}(1) + 4b_{2111}(2) &= 0, \\
2a_{211}(1) + a_{1111}(1) + 4b_{2111}(1) &= 0, \\
4a_{1111}(1) + 5b_{11111}(1) &= 0.
\end{aligned}$$

By solving the above equations we can determine inner inward flowing currents corresponding to decimation number **b=5** which is given in subsection 3.2.

Appendix III: Proof of the formula (4-6):

Here we give the proof of the formula (4-6). The number of independent Shure's invariant polynomials of degree $2k$ of 3 variables I_1, I_2, I_3 with the constraint:

$$I_1 + I_2 + I_3 = 0, \quad (\text{III-1})$$

is equal to

$$P_3(2K) - P_3(2k - 1), \quad (\text{III-2})$$

where $P_3(m)$ is the number of partition of m into at most three independent non-negative integers. If we define $M_k(n)$, the number of partitions of n into exactly k non-negative integers, then we have

$$P_3(n) = \sum_{k=1}^3 M_k(n). \quad (\text{III-3})$$

Obviously

$$M_1(2k) = M_1(2k - 1). \quad (\text{III-4})$$

and

$$M_2(2k) = M_2(2k - 1) + 1, \quad M_2(2k) = M_2(2k + 1). \quad (\text{III-5})$$

If we denote the partition of $2k$, $2k-1$ and $2k+1$ into two non-negative integers respectively by: (l_1, l_2) , (m_1, m_2) and (n_1, n_2) then in the case of $l_2 = m_2 = n_2$ we will have

$$l_1 = m_1 - 1 = n_1 + 1. \quad (\text{III-6})$$

Therefore, for all values of $l_1 > k$, there is a one to one correspondence between the $M_2(2k), M_2(2k-1)$ and $M_2(2k+1)$. Only for $l_1 = k$, n_1 can be equal to $k+1$, but m_1 cannot be equal to $k-1$, thereof the relations (III-4) and (III-5) follows. Now, we are ready to prove that

$$M_3(2k) = M_3(2k-1) + [k/3]. \quad (\text{III-7})$$

If we denote the partition of $2k$ and $2k-1$ into three non-negative integers by (l_1, l_2, l_3) and (m_1, m_2, m_3) respectively, then using the relations (III-4) and (III-5), we can prove $M_3(2k) = M_3(2k-1)$ for $m_1 = l_1 = \text{odd}$; and for $l_1 = m_1 = \text{even}$, we would have $M_3(2k) = M_3(2k-1)$.

Since l_1 takes values between 1 and $[2k/3]$, where $[[2k/3]/2] = [k/3]$ of them correspond to even values of l_1 , the relation (III-7) follows and the proof is complete.

APPENDIX IV: Solution of Constraints Over Schure's invariants polynomials:

Here in this appendix by solving the constraints over Schure's polynomials of degree 12, 14, 16, 18, 20 and 22, we have expressed the dependent invariant polynomials in terms of the independent invariant polynomials.

1) Solution of Constraints of degree 12:

The invariant polynomials S_{12} , $S_{8,2,2}$ and $S_{6,3,3}$ are considered to be independent and the other dependent invariant polynomials can be written in terms of them as follows:

$$S_{5,4,3} = -S_{6,3,3}, \quad S_{4,4,4} = \frac{S_{6,3,3}}{3}$$

$$S_{7,3,2} = -S_{8,2,2}, \quad S_{6,4,2} = S_{8,2,2} - 2S_{6,3,3},$$

$$\begin{aligned}
S_{5,5,2} &= \frac{3S_{6,3,3} - S_{8,2,2}}{2}, & S_{6,5,1} &= \frac{S_{8,2,2} - 3S_{5,3,3}}{2} \\
S_{7,4,1} &= \frac{7S_{6,3,3} - 3S_{8,2,2}}{2}, & S_{8,3,1} &= \frac{5S_{8,2,2} - 7S_{6,3,3}}{2}, \\
S_{9,2,1} &= \frac{7S_{6,3,3} - 9S_{8,2,2}}{2}, & S_{10,1,1} &= \frac{9S_{8,2,2} - 7S_{6,3,3}}{2}, \\
S_{11,1} &= -S_{12}, \\
S_{10,2} &= S_{12} + 7S_{6,3,3} - 9S_{8,2,2}, \\
S_{9,3} &= \frac{27S_{8,2,2} - 21S_{6,3,3} - 2S_{12}}{2}, \\
S_{8,4} &= \frac{28S_{6,3,3} - 32S_{8,2,2} + 2S_{12}}{2}, \\
S_{7,5} &= \frac{35S_{8,2,2} - 2S_{12} - 35S_{6,3,3}}{2}, \\
S_{6,6} &= \frac{2S_{12} + 38S_{6,3,3} - 36S_{8,2,2}}{4}.
\end{aligned}$$

2) Solution of Constraints of degree 14:

The invariant polynomials S_{14} , $S_{6,6,2}$ and $S_{5,5,4}$ are considered to be independent one and the other dependent invariant polynomials can be written in terms of them as follows:

$$\begin{aligned}
S_{10,3,1} &= 7S_{6,2,2} - 23S_{5,5,4}, \\
S_{8,5,1} &= 3S_{6,6,2} - S_{5,5,4}, \\
S_{9,4,1} &= -5S_{6,6,2} + 7S_{5,5,4}, \\
S_{8,3,3} &= -5S_{5,5,4}, & S_{7,4,3} &= 5S_{5,5,4}, \\
S_{6,5,3} &= -S_{5,5,4}, & S_{6,4,4} &= -2S_{5,5,4}, \\
S_{9,5} &= -S_{14} + 45S_{6,6,2} - 195S_{5,5,4}, \\
S_{10,4} &= S_{14} - 40S_{6,6,2} + 188S_{5,5,4}, \\
S_{11,3} &= -S_{14} + 33S_{6,6,2} - 165S_{5,5,4}, \\
S_{7,7} &= \frac{-S_{14} + 49S_{6,6,2} - 196S_{5,5,4}}{2},
\end{aligned}$$

$$S_{8,6} = S_{14} - 48S_{6,6,2} + 196S_{5,5,4},$$

$$S_{13,1} = -S_{14},$$

$$S_{8,4,2} = 2S_{6,6,2} - 6S_{5,5,4},$$

$$S_{12,2} = S_{14} - 22S_{6,6,2} + 110S_{5,5,4},$$

$$S_{9,3,2} = -2S_{6,6,2} + 16S_{5,5,4},$$

$$S_{7,5,2} = -2S_{6,6,2} + S_{5,5,4},$$

$$S_{10,2,2} = 2S_{6,6,2} - 16S_{5,5,4},$$

$$S_{7,6,1} = -S_{6,6,2},$$

$$S_{11,2,1} = -11S_{6,6,2} + 55S_{5,5,4},$$

$$S_{12,1,1} = 11S_{6,6,2} - 55S_{5,5,4}.$$

3) Solution of Constraints of degree 16: The invariant polynomials S_{16} , $S_{7,7,2}$, $S_{6,6,4}$ are considered to be independent and the other dependent invariant polynomials can be written in terms of them as follows:

$$S_{6,5,5} = 0$$

$$S_{7,5,4} = -2S_{6,6,4}, \quad S_{7,6,3} = -S_{6,6,4},$$

$$S_{8,8,4} = 2S_{6,6,4}, \quad S_{8,5,3} = 3S_{6,6,4},$$

$$S_{8,7,1} = -S_{7,7,2}, \quad S_{8,6,2} = -2S_{7,7,2} + S_{6,6,4},$$

$$S_{9,4,3} = -7S_{6,6,4}, \quad S_{9,5,2} = 2S_{7,7,2} - 4S_{6,6,4},$$

$$S_{9,6,1} = 3S_{7,7,2} - S_{6,6,4}, \quad S_{10,3,3} = 7S_{6,6,4},$$

$$S_{10,4,2} = -2S_{7,7,2} + 11S_{6,6,4},$$

$$S_{10,5,1} = -5S_{7,7,2} + 5S_{6,6,4},$$

$$S_{11,3,2} = 2S_{7,7,2} - 25S_{6,6,4},$$

$$S_{11,4,1} = 7S_{7,7,2} - 16S_{6,6,4},$$

$$S_{12,2,2} = -2S_{7,7,2} + 25S_{6,6,4},$$

$$S_{12,3,1} = -9S_{7,7,2} + 41S_{6,6,4},$$

$$S_{13,2,1} = 13S_{7,7,2} - 91S_{6,6,4},$$

$$S_{14,1,1} = -13S_{7,7,2} + 91S_{6,6,4},$$

$$S_{15,1} = -S_{16},$$

$$S_{14,2} = S_{16} + 26S_{7,7,2} - 182S_{6,6,4},$$

$$S_{13,3} = -S_{16} - 39S_{7,7,2} + 273S_{6,6,4},$$

$$S_{12,4} = S_{16} + 48S_{7,7,2} - 314S_{6,6,4},$$

$$S_{11,5} = -S_{16} - 55S_{7,7,2} + 330S_{6,6,4},$$

$$S_{10,6} = S_{16} + 60S_{7,7,2} - 335S_{6,6,4},$$

$$S_{9,7} = -S_{16} - 63S_{7,7,2} + 336S_{6,6,4},$$

$$S_{8,8} = \frac{S_{16} + 64S_{7,7,2} - 336S_{6,6,4}}{2}.$$

4) Solution of Constraints of degree 18:

The invariant polynomials S_{18} , $S_{14,2,2}$, $S_{7,7,4}$, $S_{8,5,5}$ are considered to be independent and the other dependent invariant polynomials can be written in terms of them as follows:

$$S_{7,6,5} = -S_{8,5,5}, \quad S_{6,6,6} = \frac{S_{8,5,5}}{3},$$

$$S_{8,7,3} = -S_{7,7,4}, \quad S_{8,6,4} = S_{8,5,5} - 2S_{7,7,4},$$

$$S_{9,5,4} = -3S_{8,5,5} + 2S_{7,7,4}, \quad S_{9,6,3} = 3S_{7,7,4} - S_{8,5,5},$$

$$S_{10,5,3} = 4S_{8,5,5} - 5S_{7,7,4}, \quad S_{10,4,4} = 3S_{8,5,5} - 2S_{7,7,4},$$

$$S_{11,4,3} = 9S_{7,7,4} - 10S_{8,5,5}, \quad S_{12,3,3} = 10S_{8,5,5} - 9S_{7,7,4},$$

$$S_{13,3,2} = -S_{14,2,2},$$

$$S_{12,4,2} = S_{14,2,2} - 20S_{8,5,5} + 18S_{7,7,4},$$

$$S_{11,5,2} = -S_{14,2,2} + 30S_{8,5,5} - 27S_{7,7,4},$$

$$S_{10,6,2} = 32S_{7,7,4} - 34S_{8,5,5} + S_{14,2,2},$$

$$S_{9,7,2} = 35S_{8,5,5} - 35S_{7,7,4} - S_{14,2,2},$$

$$S_{8,8,2} = \frac{36S_{7,7,4} + S_{14,2,2} - 35S_{8,5,5}}{2},$$

$$S_{9,8,1} = \frac{35S_{8,5,5} - S_{14,2,2} - 36S_{7,7,4}}{2},$$

$$S_{10,7,1} = \frac{106S_{7,7,4} - 105S_{8,5,5} + 3S_{14,2,2}}{2},$$

$$S_{11,6,1} = \frac{173S_{8,5,5} - 170S_{7,7,4} - 5S_{14,2,2}}{2},$$

$$S_{12,5,1} = \frac{224S_{7,7,4} - 233S_{8,5,5} + 7S_{14,2,2}}{2},$$

$$S_{13,4,1} = \frac{273S_{8,5,5} - 260S_{7,7,4} - 9S_{14,2,2}}{2},$$

$$S_{14,3,1} = 260S_{7,7,4} - 273S_{8,5,5} + 11S_{14,2,2},$$

$$S_{15,2,1} = \frac{273S_{8,5,5} - 260S_{7,7,4} - 15S_{14,2,2}}{2},$$

$$S_{16,1,1} = \frac{260S_{7,7,4} + 15S_{14,2,2} - 273S_{8,5,5}}{2},$$

$$S_{17,1} = -S_{18},$$

$$S_{16,2} = S_{18} - 260S_{7,7,4} - 15S_{14,2,2} + 273S_{8,5,5},$$

$$S_{15,3} = \frac{780S_{7,7,4} - 2S_{18} + 45S_{14,2,2} - 819S_{8,5,5}}{2},$$

$$S_{14,4} = \frac{1092S_{8,5,5} - 56S_{14,2,2} + 2S_{18} - 1040S_{7,7,4}}{2},$$

$$S_{13,5} = \frac{65S_{14,2,2} - 2S_{18} - 1365S_{8,5,5} + 1300S_{7,7,4}}{2},$$

$$S_{12,6} = \frac{2S_{18} + 1598S_{8,5,5} - 1524S_{7,7,4} - 72S_{14,2,2}}{2},$$

$$S_{11,7} = \frac{1694S_{7,7,4} + 77S_{14,2,2} - 2S_{18} - 1771S_{8,5,5}}{2},$$

$$S_{10,8} = \frac{2S_{18} + 1876S_{8,5,5} - 80S_{14,2,2} - 1800S_{7,7,4}}{2},$$

$$S_{9,9} = \frac{81S_{14,2,2} + 1836S_{7,7,4} - 2S_{18} - 1911S_{8,5,5}}{4}.$$

5) Solution of Constraints of degree 20:

The invariant polynomials S_{20} , $S_{16,2,2}$, $S_{8,8,4}$ and $S_{7,7,6}$ are considered to be independent and the other dependent invariant polynomials can be written in terms of them as follows:

$$S_{8,8,6} = -2S_{7,7,6}, \quad S_{8,7,5} = -S_{7,7,6},$$

$$S_{9,6,5} = 5S_{7,7,6}, \quad S_{9,8,3} = -S_{8,8,4},$$

$$S_{9,7,4} = S_{7,7,6} - 2S_{8,8,4}, \quad S_{10,5,5} = -5S_{7,7,6},$$

$$S_{10,6,4} = -6S_{7,7,6} + 2S_{8,8,4}, \quad S_{10,7,3} = 3S_{8,8,4} - S_{7,7,6},$$

$$S_{11,6,3} = -5S_{8,8,4} + 7S_{7,7,6}, \quad S_{11,5,4} = +16S_{7,7,6} - 2S_{8,8,4},$$

$$S_{12,4,4} = (-16S_{7,7,6} + 2S_{8,8,4}), \quad S_{12,5,3} = -23S_{7,7,6} + 7S_{8,8,4},$$

$$S_{13,4,3} = 55S_{7,7,6} - 11S_{8,8,4}, \quad S_{14,3,3} = -55S_{7,7,6} + 11S_{8,8,4},$$

$$S_{15,3,2} = S_{16,2,2},$$

$$S_{14,4,2} = -22S_{8,8,4} + S_{16,2,2} + 110S_{7,7,6},$$

$$S_{13,5,2} = -165S_{7,7,6} - S_{16,2,2} + 33S_{8,8,4},$$

$$S_{12,6,2} = -40S_{8,8,4} + 188S_{7,7,6} + S_{16,2,2},$$

$$S_{11,7,2} = -195S_{7,7,6} + 45S_{8,8,4} - S_{16,2,2},$$

$$S_{10,8,2} = -48S_{8,8,4} + 196S_{7,7,6} + S_{16,2,2},$$

$$S_{9,9,2} = \frac{-196S_{7,7,6} + 49S_{8,8,4} - S_{16,2,2}}{2},$$

$$S_{10,9,1} = \frac{S_{16,2,2} + 196S_{7,7,6} - 49S_{8,8,4}}{2},$$

$$\begin{aligned}
S_{11,8,1} &= \frac{-588S_{7,7,6} + 145S_{8,8,4} - 3S_{16,2,2}}{2}, \\
S_{12,7,1} &= \frac{+978S_{7,7,6} - 235S_{8,8,4} + 5S_{16,2,2}}{2}, \\
S_{13,6,1} &= \frac{-1354S_{7,7,6} + 315S_{8,8,4} - 7S_{16,2,2}}{2}, \\
S_{14,5,1} &= \frac{9S_{16,2,2} - 381S_{8,8,4} + 1684S_{7,7,6}}{2}, \\
S_{15,4,1} &= \frac{425S_{8,8,4} - 1904S_{7,7,6} - 11S_{16,2,2}}{2}, \\
S_{16,3,1} &= \frac{13S_{16,2,2} + 1904S_{7,7,6} - 425S_{8,8,4}}{2}, \\
S_{17,2,1} &= \frac{-1904S_{7,7,6} + 425S_{8,8,4} - 17S_{16,2,2}}{2}, \\
S_{18,1,1} &= \frac{17S_{16,2,2} - 425S_{8,8,4} + 1904S_{7,7,6}}{2}, \\
S_{19,1} &= -S_{20},
\end{aligned}$$

$$\begin{aligned}
S_{18,2} &= S_{20} + 425S_{8,8,4} - 1904S_{7,7,6} - 17S_{16,2,2}, \\
S_{17,3} &= \frac{51S_{16,2,2} - 1275S_{8,8,4} + 5712S_{7,7,6} - 2S_{20}}{2}, \\
S_{16,4} &= \frac{2S_{20} - 64S_{16,2,2} + 1700S_{8,8,4} - 7616S_{7,7,6}}{2}, \\
S_{15,5} &= \frac{75S_{16,2,2} - 2125S_{8,8,4} - 2S_{20} + 9520S_{7,7,6}}{2}, \\
S_{14,6} &= \frac{2S_{20} + 2506S_{8,8,4} - 11204S_{7,7,6} - 84S_{16,2,2}}{2}, \\
S_{13,7} &= \frac{91S_{16,2,2} + 12558S_{7,7,6} - 2821S_{8,8,4} - 2S_{20}}{2}, \\
S_{12,8} &= \frac{2S_{20} - 96S_{16,2,2} + 3056S_{8,8,4} - 13536S_{7,7,6}}{2}, \\
S_{11,9} &= \frac{99S_{16,2,2} - 2S_{20} - 3201S_{8,8,4} + 14124S_{7,7,6}}{2}, \\
S_{10,10} &= \frac{2S_{20} - 100S_{16,2,2} + 3250S_{8,8,4} - 14320S_{7,7,6}}{2}.
\end{aligned}$$

6) Solution of Constraints of degree 22:

The invariant polynomials S_{22} , $S_{18,2,2}$, $S_{9,9,4}$ and $S_{8,8,6}$ are considered to be independent and the other dependent invariant polynomials can be written in terms of them as follows:

$$S_{8,7,7} = 0, \quad S_{9,7,6} = -2S_{8,8,6},$$

$$S_{9,8,5} = -S_{8,8,6}, \quad S_{10,6,6} = 2S_{8,8,6},$$

$$S_{10,7,5} = 3S_{8,8,6}, \quad S_{10,9,3} = -S_{9,9,4},$$

$$S_{10,8,4} = S_{8,8,6} - 2S_{9,9,4}, \quad S_{11,6,5} = -7S_{8,8,6},$$

$$S_{11,7,4} = 2S_{9,9,4} - 4S_{8,8,6},$$

$$S_{11,8,3} = 3S_{9,9,4} - S_{8,8,6}, \quad S_{12,5,5} = 7S_{8,8,6},$$

$$S_{12,6,4} = 11S_{8,8,6} + 2S_{9,9,4}, \quad S_{12,7,3} = 5S_{8,8,6} - 5S_{9,9,4},$$

$$S_{13,5,4} = 2S_{9,9,4} - 25S_{8,8,6}, \quad S_{13,6,3} = 7S_{9,9,4} - 16S_{8,8,6},$$

$$S_{14,4,4} = 25S_{8,8,6} - 2S_{9,9,4}, \quad S_{14,5,3} = 41S_{8,8,6} - 9S_{9,9,4},$$

$$S_{15,4,3} = 13S_{9,9,4} - 91S_{8,8,6}, \quad S_{16,3,3} = 91S_{8,8,6} - 13S_{9,9,4},$$

$$S_{17,3,2} = -S_{18,2,2},$$

$$S_{16,4,2} = 26S_{9,9,4} - 182S_{8,8,6} + S_{18,2,2},$$

$$S_{15,5,2} = 273S_{8,8,6} - 39S_{9,9,4} - S_{18,2,2},$$

$$S_{14,6,2} = 48S_{9,9,4} - 314S_{8,8,6} + S_{18,2,2},$$

$$S_{13,7,2} = 330S_{8,8,6} - 55S_{9,9,4} - S_{18,2,2},$$

$$S_{12,8,2} = 60S_{9,9,4} + S_{18,2,2} - 335S_{8,8,6},$$

$$S_{11,9,2} = 336S_{8,8,6} - 63S_{9,9,4} - S_{18,2,2},$$

$$S_{10,10,2} = \frac{S_{18,2,2} + 64S_{9,9,4} - 336S_{8,8,6}}{2},$$

$$S_{11,10,1} = \frac{-64S_{9,9,4} + 336S_{8,8,6} - S_{18,2,2}}{2},$$

$$\begin{aligned}
S_{12,9,1} &= \frac{190S_{9,9,4} + 3S_{18,2,2} - 1008S_{8,8,6}}{2}, \\
S_{13,8,1} &= \frac{-310S_{9,9,4} - 5S_{18,2,2} + 1678S_{8,8,6}}{2}, \\
S_{14,7,1} &= \frac{-2338S_{8,8,6} + 7S_{18,2,2} + 420S_{9,9,4}}{2}, \\
S_{15,6,1} &= \frac{2966S_{8,8,6} - 9S_{18,2,2} - 516S_{9,9,4}}{2}, \\
S_{16,5,1} &= \frac{11S_{18,2,2} + 594S_{9,9,4} - 3512S_{8,8,6}}{2}, \\
S_{17,4,1} &= \frac{3876S_{8,8,6} - 13S_{18,2,2} - 646S_{9,9,4}}{2}, \\
S_{18,3,1} &= \frac{-3876S_{8,8,6} + 646S_{9,9,4} + 15S_{18,2,2}}{2}, \\
S_{19,2,1} &= \frac{-646S_{9,9,4} - 19S_{18,2,2} + 3876S_{8,8,6}}{2}, \\
S_{20,1,1} &= \frac{-3876S_{8,8,6} + 19S_{18,2,2} + 646S_{9,9,4}}{2}, \\
S_{21,1} &= -S_{22}, \\
S_{20,2} &= S_{22} - 19S_{18,2,2} - 646S_{9,9,4} + 3876S_{8,8,6}, \\
S_{19,3} &= \frac{57S_{18,2,2} - 2S_{22} + 1938S_{9,9,4} - 11628S_{8,8,6}}{2}, \\
S_{18,4} &= \frac{2S_{22} - 72S_{18,2,2} - 2584S_{9,9,4} + 15504S_{8,8,6}}{2}, \\
S_{17,5} &= \frac{3230S_{9,9,4} + 85S_{18,2,2} - 19380S_{8,8,6} - 2S_{22}}{2}, \\
S_{16,6} &= \frac{2S_{22} - 96S_{18,2,2} - 3824S_{9,9,4} + 22892S_{8,8,6}}{2}, \\
S_{15,7} &= \frac{105S_{18,2,2} + 4340S_{9,9,4} - 25858S_{8,8,6} - 2S_{22}}{2}, \\
S_{14,8} &= \frac{28196S_{8,8,6} - 112S_{18,2,2} + 2S_{22} - 4760S_{9,9,4}}{2}, \\
S_{13,9} &= \frac{5070S_{9,9,4} + 117S_{18,2,2} - 29874S_{8,8,6} - 2S_{22}}{2}, \\
S_{12,10} &= \frac{30882S_{8,8,6} + 2S_{22} - 120S_{18,2,2} - 5260S_{9,9,4}}{2}, \\
S_{11,11} &= \frac{121S_{18,2,2} + 4678S_{9,9,4} - 2S_{22} - 31218S_{8,8,6}}{4}.
\end{aligned}$$

APPENDIX IV:the recursion relation and λ_{max} for $q \geq 6$ a)b=2 q=6

$$\begin{pmatrix} A_{6(n+1)} \\ A_{4,1,1(n+1)} \end{pmatrix} = \begin{pmatrix} 1.090534979 & -.1646090535 \\ .1481481481 & .1851851852 \end{pmatrix} \begin{pmatrix} A_{6(n)} \\ A_{4,1,1(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.062745991.$$

q=8

$$\begin{pmatrix} A_{8(n+1)} \\ A_{3,3,2(n+1)} \end{pmatrix} = \begin{pmatrix} 1.039323274 & .5121170553 \\ -.03840877915 & .1742112483 \end{pmatrix} \begin{pmatrix} A_{8(n)} \\ A_{3,3,2(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.015955376.$$

q=10

$$\begin{pmatrix} A_{10(n+1)} \\ A_{4,4,2(n+1)} \end{pmatrix} = \begin{pmatrix} 1.017375400 & -.3840877915 \\ .02987349489 & .1486053955 \end{pmatrix} \begin{pmatrix} A_{10(n)} \\ A_{4,4,2(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.003961045.$$

q=12

$$\begin{pmatrix} A_{12(n+1)} \\ A_{8,2,2(n+1)} \\ A_{6,3,3(n+1)} \end{pmatrix} = \begin{pmatrix} 1.007711110 & -.1266744568 & .1217068310 \\ .05378583888 & .1908753747 & -.1417805551 \\ .007315957933 & .07681755830 & -.04709647920 \end{pmatrix} \begin{pmatrix} A_{12(n)} \\ A_{8,2,2(n)} \\ A_{6,3,3(n+1)} \end{pmatrix},$$

$$\lambda_{max} = 1.000983621.$$

q=14

$$\begin{pmatrix} A_{14(n+1)} \\ A_{6,6,2(n+1)} \\ A_{5,5,4(n+1)} \end{pmatrix} = \begin{pmatrix} 1.003425906 & -.1557074696 & .6579896295 \\ .01048470103 & -.09541040304 & .5668362057 \\ -.001896729835 & -.04335382479 & .2332808346 \end{pmatrix} \begin{pmatrix} A_{14(n)} \\ A_{6,6,2(n)} \\ A_{5,5,4(n+1)} \end{pmatrix},$$

$$\lambda_{max} = 1.000244974.$$

q=16

$$\begin{pmatrix} A_{16(n+1)} \\ A_{7,7,2(n+1)} \\ A_{6,6,4(n+1)} \end{pmatrix} = \begin{pmatrix} 1.001522485 & .09132402907 & -.5053262942 \\ -.006679533152 & -.1334905160 & 1.056414123 \\ .001475234316 & -.03567658498 & .2661499583 \end{pmatrix} \begin{pmatrix} A_{16(n)} \\ A_{7,7,2(n)} \\ A_{6,6,4(n+1)} \end{pmatrix},$$

$$\lambda_{max} = 1.000061131.$$

q=18

$$\begin{pmatrix} A_{18(n+1)} \\ A_{14,2,2(n+1)} \\ A_{7,7,4(n+1)} \\ A_{8,5,5(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000676645 & -.02587988061 & -.5694995445 & .5946011699 \\ .04976341868 & .3702685508 & 6.165437409 & -6.474053436 \\ -.0007861225946 & -.004847198466 & -.1026962541 & .1087868226 \\ .0003612818732 & .009483649173 & .1320936849 & -.1378233896 \\ 1.000015270 & .1122123554 & .01717707276 & .001020854222 \end{pmatrix} \begin{pmatrix} A_{18(n)} \\ A_{14,2,2(n)} \\ A_{7,7,4(n)} \\ A_{8,5,5(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.000015270.$$

q=20

$$\begin{pmatrix} A_{20(n+1)} \\ A_{16,2,2(n+1)} \\ A_{8,8,4(n+1)} \\ A_{7,7,6(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000300729 & -.01428439338 & .4535545013 & -2.007507524 \\ .04955188625 & .4249966010 & -10.27981253 & 46.05861448 \\ .0005177630138 & .0009212316079 & -.05329922806 & .2455943625 \\ -.00009366567084 & -.002585841556 & .05473587640 & -.2437584270 \end{pmatrix} \begin{pmatrix} A_{20(n)} \\ A_{16,2,2(n)} \\ A_{8,8,4(n)} \\ A_{7,7,6(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.000003815.$$

q=22

$$\begin{pmatrix} A_{22(n+1)} \\ A_{18,2,2(n+1)} \\ A_{9,9,4(n+1)} \\ A_{8,8,6(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000133657 & -.007718672824 & -.3331935728 & 1.963553834 \\ .04945789936 & .4775535736 & 15.78748967 & -94.73506749 \\ -.0003298534890 & .001784479705 & .02202297308 & -.1217001716 \\ .00007285107732 & .002090751581 & .06241497465 & -.3728127167 \end{pmatrix} \begin{pmatrix} A_{22(n)} \\ A_{18,2,2(n)} \\ A_{9,9,4(n)} \\ A_{8,8,6(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.000000954.$$

b)b=3 q=6

$$\begin{pmatrix} A_{6(n+1)} \\ A_{4,1,1(n+1)} \end{pmatrix} = \begin{pmatrix} 1.082584637 & -.07343878826 \\ .1808430161 & .04755671531 \end{pmatrix} \begin{pmatrix} A_{6(n)} \\ A_{4,1,1(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.069590059.$$

q=8

$$\begin{pmatrix} A_{8(n+1)} \\ A_{3,3,2(n+1)} \end{pmatrix} = \begin{pmatrix} 1.025056713 & .1818484281 \\ -.04533096632 & .03647844913 \end{pmatrix} \begin{pmatrix} A_{8(n)} \\ A_{3,3,2(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.016646559.$$

q=10

$$\begin{pmatrix} A_{10(n+1)} \\ A_{4,4,2(n+1)} \end{pmatrix} = \begin{pmatrix} 1.007845912 & -.1085523780 \\ .03420274178 & .02815706873 \end{pmatrix} \begin{pmatrix} A_{10(n)} \\ A_{4,4,2(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.004041374.$$

q=12

$$\begin{pmatrix} A_{12(n+1)} \\ A_{8,2,2(n+1)} \\ A_{6,3,3(n+1)} \end{pmatrix} = \begin{pmatrix} 1.002501314 & -.02774717658 & .02388825224 \\ .06022405661 & .05660671065 & -.04390015579 \\ .01079114607 & .01664662055 & -.01252511118 \end{pmatrix} \begin{pmatrix} A_{12(n)} \\ A_{8,2,2(n)} \\ A_{6,3,3(n+1)} \end{pmatrix},$$

$$\lambda_{max} = 1.001026135.$$

q=14

$$\begin{pmatrix} A_{14(n+1)} \\ A_{6,6,2(n+1)} \\ A_{5,5,4(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000805713 & -.02587192638 & .1198110012 \\ .008633787717 & -.02491545222 & .1279091010 \\ -.002716116832 & -.009331353906 & .04723947173 \end{pmatrix} \begin{pmatrix} A_{14(n)} \\ A_{6,6,2(n)} \\ A_{5,5,4(n+1)} \end{pmatrix},$$

$$\lambda_{max} = 1.000246359.$$

q=16

$$\begin{pmatrix} A_{16(n+1)} \\ A_{7,7,2(n+1)} \\ A_{6,6,4(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000261069 & .01135511506 & -.07104143816 \\ -.004468408282 & -.03190697902 & .2276301973 \\ .002050937173 & -.007546589748 & .05339121346 \end{pmatrix} \begin{pmatrix} A_{16(n)} \\ A_{7,7,2(n)} \\ A_{6,6,4(n+1)} \end{pmatrix},$$

$$\lambda_{max} = 1.000061317.$$

q=18

$$\begin{pmatrix} A_{18(n+1)} \\ A_{14,2,2(n+1)} \\ A_{7,7,4(n+1)} \\ A_{8,5,5(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000084877 & -.002388883753 & -.04740112573 & .04969629320 \\ .06005002155 & .07399773075 & 1.274170117 & -1.337882220 \\ -.0009014749149 & -.0004616094519 & -.008788472030 & .009233751643 \\ .0006471907605 & .002494396769 & .04199894504 & -.04409378714 \end{pmatrix} \begin{pmatrix} A_{18(n)} \\ A_{1422(n)} \\ A_{7,7,4(n)} \\ A_{8,5,5(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.000015297.$$

q=20

$$\begin{pmatrix} A_{20(n+1)} \\ A_{16,2,2(n+1)} \\ A_{8,8,4(n+1)} \\ A_{7,7,6(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000027647 & -.0009744218073 & .02806977909 & -.1253211808 \\ .05999917318 & .08477443355 & -2.107893057 & 9.443400292 \\ .0005178070963 & -.0004282216637 & .009448647802 & -.04228755067 \\ -.0001628983371 & -.0006696634259 & .01635654929 & -.07326845888 \end{pmatrix} \begin{pmatrix} A_{20(n)} \\ A_{16,2,2(n)} \\ A_{8,8,4(n)} \\ A_{7,7,6(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.000003819.$$

q=22

$$\begin{pmatrix} A_{22(n+1)} \\ A_{18,2,2(n+1)} \\ A_{9,9,4(n+1)} \\ A_{8,8,6(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000006761 & -.0002279253840 & -.009315026611 & .05551458536 \\ .05998283956 & .09512343412 & 3.219304290 & -19.31589634 \\ -.0002679923170 & .0009873502040 & .03199666409 & -.1919144288 \\ .0001230048643 & .0005306869789 & .01770755556 & -.1062348858 \end{pmatrix} \begin{pmatrix} A_{22(n)} \\ A_{18,2,2(n)} \\ A_{9,9,4(n)} \\ A_{8,8,6(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.000002379.$$

c) **b=4 q=6**

$$\begin{pmatrix} A_{6(n+1)} \\ A_{4,1,1(n+1)} \end{pmatrix} = \begin{pmatrix} 1.076310660 & -.03365228885 \\ .1889456076 & .01616947736 \end{pmatrix} \begin{pmatrix} A_{6(n)} \\ A_{4,1,1(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.070278597.$$

q=8

$$\begin{pmatrix} A_{8(n+1)} \\ A_{3,3,2(n+1)} \end{pmatrix} = \begin{pmatrix} 1.019524120 & .06535850011 \\ -.05434716964 & 371.8425102 \end{pmatrix} \begin{pmatrix} A_{8(n)} \\ A_{3,3,2(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.0195338.$$

q=10

$$\begin{pmatrix} A_{10(n+1)} \\ A_{4,4,2(n+1)} \end{pmatrix} = \begin{pmatrix} 1.005151073 & -.03282144763 \\ .03495279823 & .007384841623 \end{pmatrix} \begin{pmatrix} A_{10(n)} \\ A_{4,4,2(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.003999975.$$

q=12

$$\begin{pmatrix} A_{12(n+1)} \\ A_{8,2,2(n+1)} \\ A_{6,3,3(n+1)} \end{pmatrix} = \begin{pmatrix} 1.001388320 & -.007332723661 & .006026327075 \\ .06394351082 & .009657753638 & -.007508055989 \\ .01147176558 & .004507477340 & -.003470225654 \end{pmatrix} \begin{pmatrix} A_{12(n)} \\ A_{8,2,2(n)} \\ A_{6,3,3(n+1)} \end{pmatrix},$$

$$\lambda_{max} = 1.000986557.$$

q=14

$$\begin{pmatrix} A_{14(n+1)} \\ A_{6,6,2(n+1)} \\ A_{5,5,4(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000380234 & -.006079338806 & .02938006638 \\ .008215782561 & -.006916822481 & .03482160520 \\ -.002872694778 & -.002513285925 & .01260695484 \end{pmatrix} \begin{pmatrix} A_{14(n)} \\ A_{6,6,2(n)} \\ A_{5,5,4(n+1)} \end{pmatrix},$$

$$\lambda_{max} = 1.000245203.$$

q=16

$$\begin{pmatrix} A_{16(n+1)} \\ A_{7,7,2(n+1)} \\ A_{6,6,4(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000105428 & .002387138955 & -.01597611501 \\ -.004013610557 & -.008712843003 & .06128427304 \\ .002158295483 & -.002024493055 & .01421013434 \end{pmatrix} \begin{pmatrix} A_{16(n)} \\ A_{7,7,2(n)} \\ A_{6,6,4(n+1)} \end{pmatrix},$$

$$\lambda_{max} = 1.000061148.$$

q=18

$$\begin{pmatrix} A_{18(n+1)} \\ A_{14,2,2(n+1)} \\ A_{7,7,4(n+1)} \\ A_{8,5,5(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000029509 & -.0004497502350 & -.008243519261 & .008652383249 \\ .06187422841 & .01967111602 & .3403871980 & -.3574066079 \\ -.0009123885980 & -.00009719487958 & -.001741086261 & .001828250148 \\ .0007092075724 & .0006921244620 & .01190983853 & -.01250523720 \end{pmatrix} \begin{pmatrix} A_{18(n)} \\ A_{14,2,2(n)} \\ A_{7,7,4(n)} \\ A_{8,5,5(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.000015271.$$

q=20

$$\begin{pmatrix} A_{20(n+1)} \\ A_{16,2,2(n+1)} \\ A_{8,8,4(n+1)} \\ A_{7,7,6(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000008319 & -.0001645061367 & .004348107103 & -.01946561802 \\ .06184409730 & .02252802695 & -.5624149410 & 2.519619600 \\ .0005078921879 & -.0001439100811 & .003508323524 & -.01571652660 \\ -.0001776181030 & -.0001853563027 & .004606804705 & -.02063832629 \end{pmatrix} \begin{pmatrix} A_{20(n)} \\ A_{16,2,2(n)} \\ A_{8,8,4(n)} \\ A_{7,7,6(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.000003788.$$

q=22

$$\begin{pmatrix} A_{22(n+1)} \\ A_{18,2,2(n+1)} \\ A_{9,9,4(n+1)} \\ A_{8,8,6(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000002358 & -.00005834048617 & -.002108427732 & .01263785041 \\ .06183577589 & .02527262136 & .8582479458 & -.5149488875 \\ -.0002481471736 & .0002926869789 & .009839879848 & -.05903810198 \\ .0001334512767 & .0001464026161 & .004954070521 & -.02972423554 \end{pmatrix} \begin{pmatrix} A_{22(n)} \\ A_{18,2,2(n)} \\ A_{9,9,4(n)} \\ A_{8,8,6(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.000000954.$$

d)**b=5 q=6**

$$\begin{pmatrix} A_{6(n+1)} \\ A_{4,1,1(n+1)} \end{pmatrix} = \begin{pmatrix} 1.044360741 & -.01006023599 \\ .3156061304 & .003157528970 \end{pmatrix} \begin{pmatrix} A_{6(n)} \\ A_{4,1,1(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.041302331.$$

q=8

$$\begin{pmatrix} A_{8(n+1)} \\ A_{3,3,2(n+1)} \end{pmatrix} = \begin{pmatrix} 1.009037038 & .01421998195 \\ .07691145227 & -.005139564359 \end{pmatrix} \begin{pmatrix} A_{8(n)} \\ A_{3,3,2(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.010114286.$$

q=10

$$\begin{pmatrix} A_{10(n+1)} \\ A_{4,4,2(n+1)} \end{pmatrix} = \begin{pmatrix} 1.001897065 & -.005511338227 \\ .03479355808 & .002253185745 \end{pmatrix} \begin{pmatrix} A_{10(n)} \\ A_{4,4,2(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.001705201.$$

q=12

$$\begin{pmatrix} A_{12(n+1)} \\ A_{8,2,2(n+1)} \\ A_{6,3,3(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000405841 & -.0009777920845 & .0007888947248 \\ .06389106764 & .002988237303 & -.002324086713 \\ .01925523989 & .001402362796 & -.001089147707 \end{pmatrix} \begin{pmatrix} A_{12(n)} \\ A_{8,2,2(n)} \\ A_{6,3,3(n+1)} \end{pmatrix},$$

$$\lambda_{max} = 1.000358489.$$

q=14

$$\begin{pmatrix} A_{14(n+1)} \\ A_{6,6,2(n+1)} \\ A_{5,5,4(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000087998 & -.0006533768126 & .003203625771 \\ .008004412267 & -.002154118594 & .01079323997 \\ .004778768070 & .001472733806 & -.007372115949 \end{pmatrix} \begin{pmatrix} A_{14(n)} \\ A_{6,6,2(n)} \\ A_{5,5,4(n+1)} \end{pmatrix},$$

$$\lambda_{max} = 1.000097981.$$

q=16

$$\begin{pmatrix} A_{16(n+1)} \\ A_{7,7,2(n+1)} \\ A_{6,6,4(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000019275 & .0002082580899 & -.001424787973 \\ .003961084604 & .002614693987 & -.01832747999 \\ .002150223007 & -.0006222811611 & .004359625253 \end{pmatrix} \begin{pmatrix} A_{16(n)} \\ A_{7,7,2(n)} \\ A_{6,6,4(n+1)} \end{pmatrix},$$

$$\lambda_{max} = 1.000017019.$$

q=18

$$\begin{pmatrix} A_{18(n+1)} \\ A_{1422(n+1)} \\ A_{774(n+1)} \\ A_{855(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000004256 & -.00003193265577 & -.0005685434366 & .0005968699293 \\ .06193405435 & .006096146153 & .1056112005 & -.1108917618 \\ .001017169377 & .00006930842110 & .001204750315 & -.001264989081 \\ .001191712555 & .0002627970450 & .004548661710 & -.004776093729 \end{pmatrix} \begin{pmatrix} A_{18(n)} \\ A_{14,2,2(n)} \\ A_{7,7,4(n)} \\ A_{855(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.000002408.$$

q=20

$$\begin{pmatrix} A_{20(n+1)} \\ A_{16,2,2(n+1)} \\ A_{8,8,4(n+1)} \\ A_{7,7,6(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000000946 & -.000009482285099 & .0002434259750 & -.001090262082 \\ .06190933065 & .006979673785 & -.1744166676 & .7813866910 \\ .0004953876882 & -.00004636993771 & .001150699865 & -.005155112958 \\ .0002958221549 & .0001041331110 & -.002597984312 & .01163895822 \end{pmatrix} \begin{pmatrix} A_{20(n)} \\ A_{16,2,2(n)} \\ A_{8,8,4(n)} \\ A_{7,7,6(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.000000141.$$

q=22

$$\begin{pmatrix} A_{22(n+1)} \\ A_{18,2,2(n+1)} \\ A_{9,9,4(n+1)} \\ A_{8,8,6(n+1)} \end{pmatrix} = \begin{pmatrix} 1.000000211 & -.000002740626906 & -.00009573710813 & .0005742469550 \\ .06190294246 & .007829018982 & .2660888394 & -1.596533072 \\ .0002452104252 & -.00008750002217 & .2660888394 & .01778872116 \\ .0001330977107 & .00004502739760 & .001528690467 & -.009172137322 \end{pmatrix} \begin{pmatrix} A_{22(n)} \\ A_{18,2,2(n)} \\ A_{9,9,4(n)} \\ A_{8,8,6(n)} \end{pmatrix},$$

$$\lambda_{max} = 1.000000086.$$

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Figures Captions

Figur 1. Sierpinsky fractal with decimation number $b=3$, partitions of 2 denote the subfractals and partitions of 3 indicate the vertices , respectively.

Figure 2. Sierpinski fractal resistor networks with decimation number $b=3$.

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